NOTATION

t, temperature; r, z, radial and axial coordinates; 2h, thickness of the inclusion; and $\delta(\xi)$, Dirac delta function.

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NUMERICAL-ANALYTICAL METHOD OF SOLVING THE NONLINEAR HEAT-CONDUCTION PROBLEM FOR A DOUBLY CONNECTED VARIABLE-THICKNESS PLATE

A. I. Uzdalev and E. N. Bryukhanova

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The method permits construction of an approximate solution in an analytically closed form on each of the radial rays into which the plate domain is separated.

1. We have an isotropic plate whose external L_1 and internal L_2 contours are described by equations in a dimensionless polar coordinate system r, Θ:

$$r = r_\nu [1 + \varepsilon_\nu (\cos m_\nu \Theta + a_\nu \cos 2m_\nu \Theta)], \quad \nu = 1,\ 2, \tag{1}$$

where $\nu = 1$ corresponds to the external and $\nu = 2$ to the internal contours, r_ν, ε_ν, m_ν, a_ν are parameters, and $|\varepsilon_\nu| < 1$, $|a_\nu| < 1$, $r_\nu < 1$.

The plate thickness $h(r, \Theta)$ varies according to the law

$$h(r,\ \Theta) = H(r) P(\Theta), \tag{2}$$

where $H(r)$ and $P(\Theta)$ are given functions.

Boundary conditions of the first kind are satisfied on the side surfaces, i.e.,

$$T = T_\nu(\Theta) \text{ on } L_\nu,\ \nu = 1,\ 2. \tag{3}$$

Here the period of the function $T_\nu(\Theta)$ equals $2\pi/k_0$, where k_0 is a positive integer.

Let us consider the foundation of the plate heat insulated. The thermal characteristics of the material depend on the temperature.

The heat-conduction differential equation for a function of the temperature T has the form [1]

$$\frac{\partial}{\partial r}\left(rh\lambda \frac{\partial T}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \Theta}\left(h\lambda \frac{\partial T}{\partial \Theta}\right) = 0. \tag{4}$$

Here $\lambda = \lambda(T)$ is the heat-conduction coefficient.

Introducing the Kirchhoff variable

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$$\tau = \int_0^T \lambda(\xi)\, d\xi, \tag{5}$$

we represent (4) in such a manner:

$$\frac{\partial}{\partial r}\left(rh\frac{\partial \tau}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \Theta}\left(h\frac{\partial \tau}{\partial \Theta}\right) = 0. \tag{6}$$

We seek the function $\tau(r, \Theta)$ in the form

$$\tau(r, \Theta) = f(r)\, y(\Theta)[P(\Theta)]^{-1/2}.$$

By using the method of separation of variables, we obtain an equation to determine the two unknown functions $y(\Theta)$ and $f(r)$:

$$y''(\Theta) + [\alpha + \Phi(\Theta)]\, y(\Theta) = 0, \tag{7}$$

$$r^2 f''(r) + r\Psi(r) f'(r) - \alpha f(r) = 0, \tag{8}$$

where

$$\Phi(\Theta) = -\frac{(P^{1/2})''}{P^{1/2}}, \quad \Psi(r) = 1 + r\frac{H'(r)}{H(r)},$$

and α is a parameter playing the part of the eigennumber.

Therefore, the heat-conduction problem is reduced to the integration of (7) and (8) and the construction of the general solution of (6) for the function $\tau(r, \Theta)$.

2. We construct the solution of (7) by the method of finite differences. We partition the range of variation of the variable Θ in the period of solution of the problem $\tilde{\Theta}$ into n equal parts with division spacing $\tilde{h} = \tilde{\Theta}/n$. Let us note that the selection of Θ depends on the law of plate thickness variation in the direction Θ, the contour shapes, and the boundary conditions of the problem.

The approximate value of the desired function $y(\Theta)$ at the i-th division point Θ_i is denoted by Y_i. Then the second derivative $y''(\Theta)$ at this same point is represented as

$$\frac{Y_{i+1} - 2Y_i + Y_{i-1}}{\tilde{h}^2}.$$

The differential equation for $y(\Theta)$ is converted into the algebraic system

$$Y_{i-1} + [\mu + F(\Theta_i)]\, Y_i + Y_{i+1} = 0, \quad i = 0, 1, 2, \ldots, n-1, \tag{9}$$

where

$$\mu = \alpha\tilde{h}^2, \quad F(\Theta_i) = \tilde{h}^2\Phi(\Theta_i) - 2.$$

The homogeneous system of equations (9) has a nonzero solution if its determinant equals zero

$$\Delta(\mu) = 0. \tag{10}$$

The number of roots μ_k of (10) is n. Corresponding to each root are n values $Y_i^{(k)}$ determined from the system (9). Here and henceforth, we assign the following meaning to the indices k and i:k correspond to the number of the root μ_k and i to the number of the division point Θ_i. Fixing the variable Θ_i for $i = 0, 1, 2, \ldots, n-1$ determines the direction of the radii r in the polar coordinate system in the area of the plate. We later designate these directions as rays. Therefore, we find the family of values $Y_i^{(k)}$, $k = 0, 1, 2, \ldots, n-1$, on the i-th ray.

The function $f_i^{(k)}(r)$ which is a solution of (8) and has the form [1]

$$f_i^{(k)}(r) = C_i^{(k)}\varphi_1^{(k)}(r) + D_i^{(k)}\varphi_2^{(k)}(r), \quad i, k = 0, 1, 2, \ldots, n-1, \tag{11}$$

where $C_i^{(k)}$ and $D_i^{(k)}$ are constants of integration, also corresponds to each root μ_k on the i-th ray.

If $\mu_0 = 0$, then

$$\varphi_1^{(0)}(r) = \int \frac{dr}{rH(r)}, \quad \varphi_2^{(0)} = 1.$$

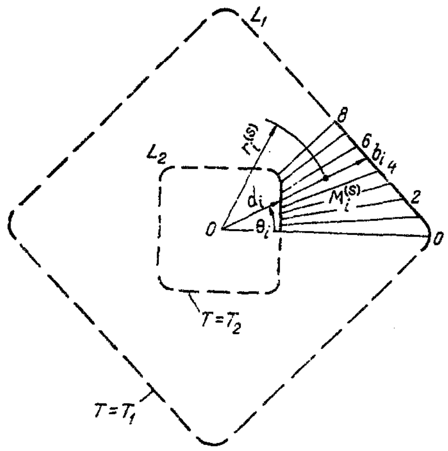

Fig. 1. Planform of a plate with rays at whose points the temperature is determined.

For $\mu_k \neq 0$

$$\varphi_j^{(k)}(r) = r^{\beta_k^{(j)}} \sum_{m=0,1,\ldots}^{\infty} a_m^{(j)} r^m,$$

$$\beta_k^{(1)} = \sqrt{\mu_k}/\tilde{h}^2, \quad \beta_k^{(2)} = -\sqrt{\mu_k}/\tilde{h}^2,$$

$$a_0^{(j)} = 1, \quad a_m^{(j)} = -\gamma_m^{(j)} \sum_{i=1}^{m} a_{m-i}^{(j)} c_i (\beta_k^{(j)} + m - i),$$

$$\gamma_m^{(j)} = [(\beta_k^{(j)} + m)(\beta_k^{(j)} + m - 1) + c_0(\beta_k^{(j)} + m) - \alpha_k]^{-1},$$

$$\alpha_k = \mu_k/\tilde{h}^2, \quad j = 1, 2, \quad k = 1, 2, \ldots, n-1.$$

The coefficients $a_m^{(j)}$ are found from the system of equations obtained by substituting (11) into (8) and equating coefficients of identical powers of r^m to zero. The constants c_i ($i = 0, 1, \ldots, m$) enter the formulas for the coefficients $a_m^{(j)}$. They are coefficients of the power series expansion of the function $\Psi(r)$. Such an expansion is possible since $r < 1$ in the area of the plate.

3. The general integral of (6) is written thus on the i-th ray:

$$\tau_i(r) = [P(\Theta_i)]^{-1/2} \sum_{k=0,1,\ldots}^{n-1} Y_i^{(k)} f_i^{(k)}(r), \quad i = 0, 1, \ldots, n-1. \tag{12}$$

Here and below the summation is over the roots of (10).

The unknown coefficients $C_i^{(k)}$ and $D_i^{(k)}$ in the solution (12) are determined from the boundary conditions of the problem (3). These latter can be converted into conditions for the function $\tau(r, \Theta)$ introduced earlier by means of (5).

The function $\tau(r, \Theta)$ satisfies the conditions

$$\tau(r, \Theta) = \int_0^{T_\nu(\Theta)} \lambda(\xi)\, d\xi \quad \text{on } L_\nu, \ \nu = 1, 2. \tag{13}$$

on the domain contours.

Let us represent the known values of the function τ on the i-th ray at points of the external contour in the form of the following sum

$$\tau(b_i, \Theta_i) = \sum_{k=0,1,\ldots}^{n-1} X_k Y_i^{(k)}, \quad i = 0, 1, \ldots, n-1. \tag{14}$$

Here b_i is the radius of the i-th ray on the external contour, X_k are newly introduced parameters. The parameters are selected in such a way that the system (14) would be satisfied identically, i.e., the system of equations (14) must be solved to determine the unknowns X_k, $k = 0, 1, \ldots, n-1$.

TABLE 1. Values of the Temperature T, °K at Points of the Plate

Θ_i	$r_i^{(1)}$	$r_i^{(2)}$	$r_i^{(3)}$	$r_i^{(4)}$	$r_i^{(5)}$	$r_i^{(6)}$
Θ_0	343	406	467	532	606	705
Θ_1	345	410	475	543	621	728
Θ_2	351	423	495	573	665	
Θ_3	359	440	523	616	731	
Θ_4	366	457	554	666		
Θ_5	372	470	579	713		
Θ_6	374	477	595	748		
Θ_7	374	478	600	764		
Θ_8	374	478	601	769		

Values of the function τ at points of the internal contour are represented analogously in terms of the parameters $Z_0, Z_1, \ldots, Z_{n-1}$

$$\tau(d_i, \Theta_i) = \sum_{k=0,1,\ldots}^{n-1} Z_k Y_i^{(k)}, \quad i = 0, 1, 2, \ldots, n-1, \tag{15}$$

which are found by solving the system (15). Here d_i is the radius of the i-th ray on the internal contour.

Furthermore, we require satisfaction of conditions (13) on the rays $i = 0, 1, \ldots, n-1$. We here use the representation of the function τ on the boundaries (14) and (15)

$$[P(\Theta_i)]^{-1/2} \sum_{k=0,1,\ldots}^{n-1} f_i^{(k)}(b_i)\, Y_i^{(k)} = \sum_{k=0,1,\ldots}^{n-1} X_k Y_i^{(k)} \quad \text{on } L_1,$$

$$[P(\Theta_i)]^{-1/2} \sum_{k=0,1,\ldots}^{n-1} f_i^{(k)}(d_i)\, Y_i^{(k)} = \sum_{k=0,1,\ldots}^{n-1} Z_k Y_i^{(k)} \quad \text{on } L_2.$$

Allowing a possible comparison between the coefficients for $Y_i^{(k)}$ on the left and right sides, we obtain a system of equations, each of which contains just two arbitrary constants of integration $\overline{C}_i^{(k)}$ and $\overline{D}_i^{(k)}$:

$$\overline{C}_i^{(k)} \varphi_1^{(k)}(b_i) + \overline{D}_i^{(k)} \varphi_2^{(k)}(b_i) = X_k, \quad i, k = 0, 1, 2, \ldots, n-1,$$
$$\overline{C}_i^{(k)} \varphi_1^{(k)}(d_i) + \overline{D}_i^{(k)} \varphi_2^{(k)}(d_i) = Z_k,$$

where

$$\overline{C}_i^{(k)} = C_i^{(k)} / \sqrt{P(\Theta_i)}, \quad \overline{D}_i^{(k)} = D_i^{(k)} / \sqrt{P(\Theta_i)}.$$

After the approximate determination of the coefficients, the general solution of (12) becomes known on any i-th ray. For a specifically given heat-conduction function $\lambda(T)$, the temperature on the rays is determined by using the relationship (5).

4. As an illustration of utilization of the numerical-analytical method of solving a nonlinear heat-conduction problem, we determine the temperature field in a plate whose boundaries are described by Eqs. (1) with the parameters

$$r_1 = 0{,}8; \quad \varepsilon_1 = 1/9; \quad m_1 = 4; \quad a_1 = 1/5;$$
$$r_2 = 0{,}3; \quad \varepsilon_2 = -1/9; \quad m_2 = 4; \quad a_2 = 1/5.$$

The plate with the mentioned parameters is a square with rounded corners and a hole of the same shape in planform (see Fig. 1). The plate thickness varies according to (2) in which $H(r) = H_0(1 - r)$, $P(\Theta) = P_0(1 + 0.25 \cos 4\Theta)$, H_0, P_0 = const.

A constant temperature $T_1 = 773$°K is maintained on the external side surface, and $T_2 = 273$°K on the internal surface. The heat-conduction coefficient is a linear function of the temperature $\lambda = \lambda_0 + \lambda_1 T$ and is governed by the parameters [3]: $\lambda_0 = 32.3$ W/(m•K), $\lambda_1 = -0.0145$ W/(m•K^2).

The period $\tilde{\Theta}$ of the solution of the problem equals $\pi/2$. By virtue of the geometric symmetry as well as the symmetry of the boundary conditions, it turns out to be sufficient to construct the solution in the area of the plate only in the range of variation of the parameter Θ between 0 and $\pi/4$.

The result of computing the values of the temperature at the points $M_i^{(s)}$ of the i-th ray with coordinates $r_i^{(s)} = d_i + 0.1s$, $s = 1, 2, \ldots$, $\Theta_i = i\tilde{h}$ is presented in Table 1.

The data in Table 1 yield a representation of the temperature distribution in the internal points of the plate area. They are obtained upon partitioning the interval $(0, \pi/4)$ into eight parts $(i = 0, 1, \ldots, 8)$ with a division spacing of $\tilde{h} = \pi/32$.

NOTATION

L_1, L_2, plate contours; r, Θ, dimensionless polar coordinates, r_ν, ε_ν, m_ν, a_ν, $\nu = 1, 2$, contour parameters; $h(r, \Theta)$, plate thickness; H, P, given functions; T_ν, $\nu = 1, 2$, value of the temperature on the L_ν contour; T, function of the temperature; λ, heat-conduction coefficient: τ, Kirchhoff variable; Φ, Ψ, known functions; α, parameter playing the part of the eigennumber; $\tilde{\Theta}$, period of the solution of the problem; n, number parts into which the interval is divided; h, division spacing; Θ_i, point of division; Y_i, an approximate value of the function $y(\Theta)$ at the division point; μ, parameter; $F(\Theta_i)$ known function; μ_k, roots of the characteristic equation; $f_i^{(k)}$, $\varphi_1^{(k)}$, $\varphi_2^{(k)}$, functions of the radius r; $C_i^{(k)}$, $D_i^{(k)}$, constants of integration; τ_i, a function of the radius r at the i-th ray; X_k, Z_k, parameters determined from the boundary conditions of the problem; $M_i^{(s)}$, a point of the i-th ray; and $r_i^{(s)}$, Θ_i, coordinates of a point on the i-th ray.

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APPROXIMATE ANALYTICAL SOLUTION OF LINEAR HEAT-CONDUCTION PROBLEMS IN REGIONS WITH NONCANONICAL BOUNDARIES

I. V. Baryshnikov and V. A. Datskovskii

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We present a method for solving linear heat-conduction problems in regions bounded by a noncanonical contour. The method is based on extending the noncanonical contour to a contour imbedded in the grid of classical coordinate systems.

The use of various modifications of the method of partial regions (see, for example, [1]) broadens the possibility of analytically solving heat-conduction problems. The main ingredient in the application of these methods is the requirement of a canonical contour bounding the computational region (it must be formed by the intersection of orthogonal coordinate surfaces of classical coordinate systems [2]).

In the present paper we offer an approximate analytical solution of linear heat-conduction problems in regions bounded by a noncanonical contour.

In connection with fields described by the Laplace equation, our method for the solution of a problem can be represented as follows: 1) a contour of complex profile bounding the computational region is extended to a contour of canonical form; 2) on the extended part of the contour a boundary condition of the second kind

$$\lambda \left.\frac{\partial T}{\partial n}\right|_s = q(s),$$

is introduced, where $q(s)$ is an unknown thermal flow distribution function on the "extended" boundary s; 3) the function $q(s)$ may be replaced by a piecewise-constant representation q_i, $i = 1, 2, \ldots, M$; 4) a solution of a field problem constructed by one of the analytical

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